Error Avoiding Quantum Codes

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Abstract The existence is proved of a class of open quantum systems that admits a linear subspace $\mathcal{C}$ of the space of states such that the restriction of the dynamical semigroup to the states built over $\mathcal{C}$ is unitary. Such subspace allows for error-avoiding (noiseless) enconding of quantum information.

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1. Introduction

In this letter we deal with the general question, in the frame of the mathematical theory of open quantum systems, whether a subset of the state space of a given open system S within the environment E exists, unaffected by the coupling of S with E. Such a challenging question raises with special enphasis in the area of *quantum computation* (QC) [1], where it finds strong motivations. QC aims to construct computational schemes, based on quantum features, more efficient (e.g. exponentially faster) than classical algorithms [2]. Quantum computation differs from classical computation in that, whereas in the latter a Turing-Boole state is specified at any time by a single integer, say n, written in binary form, the generic state $|\psi\rangle$ of a quantum computer is a superposition of states $|n\rangle$ in some appropriate Hilbert space $\mathcal{H}$, each of which can be thought of as corresponding to a classical boolean state; $|\psi\rangle = \sum_{n=00\cdots 0}^{11\cdots 1} c_n|n\rangle$. The features of $|\psi\rangle$ are described by the probability amplitudes c_n. The higher potential efficiency one may expect of quantum with respect to classical computation is ascribable just to characteristically quantum mechanical properties, such as interference (the phases of the c_n's play a role), entanglement (some of the quantum states of a complete system do not correspond to definite states of its constituting parts), von Neumann state reduction (a quantum state cannot be observed without being irreversibly disturbed), which are absent in classical computers. Moreover, quantum information processing is inherently parallel,

due to the linear structure of state space and of dynamical evolution. For example, the quantum Turing machine proposed by Deutsch [3] consists of a unitary evolution from a single initial state encoding *input* data to a final state encoding the *output*. As in Turing's scheme, the initial state encodes information on both the input and the ″program″. It is therefore clear that in the physical implementation of QC maintaining quantum coherence (namely the phase relationship between the c_n's) in any computing system is an essential requirement in order to take advantage of its specific quantum mechanical features. On the other hand any real system unavoidably interacts with some environment, wich, typically, consists of a huge amount of uncontrollable degrees of freedom. Such interaction causes a corruption of the information stored in the system as well as errors in computational steps, that may eventually lead to wrong outputs. One of the possible approaches to overcome such difficulty, in analogy with classical computation, is to resort to redundancy in encoding information, by means of the so-called (quantum) *error correcting codes* (ECC). In these schemes [4] information is encoded in linear subspaces $\mathcal{C}$ (codes) of the system Hilbert space in such a way that ″errors″ induced by the interaction with the enviroment can be detected and corrected. Of course, detection of correctable errors has to be carried over with no gain of *which-path* information about the actual system state; otherwise this would result in a further source of loss of coherence. The ECC approach appears then to aim to an *active* stabilization of quantum states by conditionally carrying on suitable quantum operations [5]. The typical system considered in the ECC literature is the *N-qubit register* R made of N replicas of a two-level system S (the qubit) where each qubit of R is assumed to be coupled with an independent environment. We shall prove here that, by relaxing the latter assumption, one can identify a class of open quantum systems which admit linear subspaces $\mathcal{C}$ such that the restriction to $\mathcal{C}$ of the dynamics is unitary. Quantum information encoded in such subspaces is therefore preserved, thus providing a strategy to maintain quantum coherence. The approach to the decoherence problem suggested by our results [6] is, in a sense, complementary to EC, in that it consists in a *passive* stabilization of quantum information. For this reason, subspaces $\mathcal{C}$ will be referred to as *Error Avoiding Codes* (*EAC*).

2. Outline

In this paper, without loss of generality [7], we shall describe the quantum dynamics of a (open) system S in terms of marginalization of the dynamics associated to a one-parameter unitary group $\{U_t\}_{t\in\mathbb{R}}$ of transformations acting on an enlarged Hilbert space (system plus *environment*). Even though this description is by no means unique, we assume that the form of the generator (Hamiltonian) of the dynamical

group is dictated by physical considerations [6]. The component of the Hamiltonian that induces a non-trivial mixing of the system and environment degrees of freedom will, as usual, be referred to as the interaction Hamiltonian H_I. In sect. 3 after defining an EAC $\mathcal{C}$ as a subspace with unitary marginal dynamics we characterize it (Lemma 3.1) by the simple property that H_I restricted to $\mathcal{C}$ should be the identity on the system space. The simplest – but physically important – example is provided by the simultaneous eigenspaces (if any) of the whole set of system operators appearing in H_I (Theorem 3.1). Such a condition can be implemented in a less trivial way by means of the reducible structure of the system Hilbert space considered as a representation space of a group $\mathcal{G}$ or of a Lie algebra $\mathcal{A}_S$ (Theorems 3.2, 3.3). In the former case $\mathcal{G}$ is required to be a symmetry group for H_I, in the latter the allowed interaction operators (*error generators*) have to belong to $\mathcal{U}(\mathcal{A}_S)$. By imposing that both module-structures are present and compatible (the representatives of elements of $\mathcal{A}_S$ are $\mathcal{G}$-invariant) one can identify (Theorem 3.4) a whole class of EAC's as the *singlet sector*; direct sum of the one-dimensional submodules of a semisimple (dynamical) Lie algebra $\mathcal{A}_S$. In sect. 4 we consider the case of a *quantum register* R defined as the collection of N replicas of a d-dimensional quantum system (*cell*) C. Assuming that the error generators $\{S_\lambda^{(i)}\}$ $(i=1,\ldots,N)$ of each cell are coupled in a replica-symmetric way to a common environment, one finds that the marginal dynamics of R is described in terms of the N-fold tensor representation ϕ_N of the *dynamical algebra* $\mathcal{A}_S$ [isomorphic to $\mathrm{sl}(d,\,\mathbb{C})$] spanned by the $S_\lambda^{(i)}$'s. Theorem 3.4 holds because ϕ_N is compatible with the natural action of the symmetric group $\mathcal{S}_N$.

3. Error Avoiding Codes

Let $\mathcal{H}_\alpha$, $\mathrm{dim}\mathcal{H}_\alpha = d_\alpha$, $(\alpha = E, S))$ be finite dimensional Hilbert spaces. The quantum system associated to $\mathcal{H}_S$ ($\mathcal{H}_E$) will be referred to as the *system* (respectively, the *environment*). The set of non-negative hermitian operators on Hilbert space $\mathcal{H}$ with trace one will be denoted by $\mathcal{S}(\mathcal{H})$; its elements will be referred to as *states*. $\mathcal{S}(\mathcal{H})$ is the convex hull of the set of *pure states*

$$\mathcal{S}_P(\mathcal{H}_S) \doteq \{\rho \in \mathcal{S}(\mathcal{H}_S) \colon \rho^2 = \rho\} \cong \mathcal{H}_S/\mathrm{U}(1)\ . \tag{1}$$

We assume the quantum system associated with $\mathcal{H}_{SE} = \mathcal{H}_S \otimes \mathcal{H}_E$ to be *closed*, *i.e.* its dynamics to be generated by a hermitian operator $H_{SE} \in \mathrm{End}(\mathcal{H}_{SE})$. The time evolution of any state $\rho \in \mathcal{S}(\mathcal{H}_{SE})$ is given by $\rho \rightarrow \rho_t \doteq U_{SE}(t)\,\rho\, U_{SE}^\dagger(t)$, where $U_{ES}(t) \doteq \exp(-i\,t\,H_{SE})$, $(t \in \mathbb{R})$ is the one-parameter unitary group generated by H_{SE}. The *marginal* dynamics on $\mathcal{H}_S$ (conditional to the initial preparation $\rho_E \in \mathcal{S}(\mathcal{H}_E)$) is given by

$$\mathcal{E}_t^{\rho_E} \colon \mathcal{S}(\mathcal{H}_S) \rightarrow \mathcal{S}(\mathcal{H}_E) \colon \rho \rightarrow \mathrm{tr}^E \left(U_{SE}(t)\,\rho \otimes \rho_E\, U_{SE}^\dagger(t) \right)\ . \tag{2}$$

The *dynamical semigrup* $\{\mathcal{E}_t\}_{t\geq 0}$ does not leave invariant the set of pure states. This a characteristic quantum phenomenon known as *decoherence*. It reflects the fact that the system-environment interaction *entangles* the degrees of freedom of S with those of E in such a way that, despite unitarity (which does indeed preserve purity of the overall joint state) each of the two subsystems has no longer a (pure) state of its own: the two subsystem have became *inseparable* [8]. From the point of view of quantum information this amounts to a corruption of the initial state.
For $\mathcal{C}$ a $d_{\mathcal{C}}$-dimensional linear subspace of $\mathcal{H}_S$, we denotes by $\mathcal{A}(\mathcal{C})$ the subalgebra of $\mathrm{End}(\mathcal{H}_S)$ leaving $\mathcal{C}$ invariant: $\mathcal{A}(\mathcal{C}) \doteq \{X \in \mathrm{End}(\mathcal{H}_S) \colon X\,\mathcal{C} \subset \mathcal{C}\}$.
DEFINITION 3.1. *A linear subspace $\mathcal{C} \neq \{0\}$ of $\mathcal{H}_S$, is an error avoiding code (EAC) iff*
i) $\exists\, H_S \in \mathcal{A}(\mathcal{C})$, H_S, hermitian, is such that, $\forall \rho_E \in \mathcal{S}(\mathcal{H}_E)$, $\rho \in \mathcal{S}(\mathcal{C}) \Rightarrow \mathcal{E}_t^{\rho_E}(\rho) = e^{-i\,t\,H_S}\,\rho\, e^{i\,t\,H_S} (\forall t \in \mathbb{R})$.
ii) $\mathcal{C}$ is maximal (i.e. it is not a proper subspace of any space for which i) holds).
Each state in $\mathcal{C}$ will be referred to as noiseless.
Definition 3.1 of *EAC* can be summarized in terms of commutativity ($\forall t \in \mathbb{R}$) of the following diagram:

$$\begin{array}{ccc}
\mathcal{S}(\mathcal{H}) & \xrightarrow{\mathcal{E}_t} & \mathcal{S}(\mathcal{H}) \\
\uparrow \iota & & \uparrow \iota \\
\mathcal{S}_P(\mathcal{C}) & \xrightarrow{\mathrm{Ad}(U_t)} & \mathcal{S}_P(\mathcal{C}) \\
\uparrow \cong & & \uparrow \cong \\
\mathcal{C}/\mathrm{U}(1) & \xrightarrow{U_t} & \mathcal{C}/\mathrm{U}(1)
\end{array}$$

Here $\cong$ is the isomorphism defined in equation (1) and ι is the canonical inclusion map.
Remark 1. The eigenstates of H_S in $\mathcal{C}$ are stationary states (i.e. $\mathcal{E}_t(\rho) = \rho\,, \forall t \in \mathbb{R}$). $\mathcal{C} \neq \{0\}$ means that there exists a set of initial preparations for which no information loss occurs. Since the minimal system which permits useful enconding of quantum information is a two-level system (*qubit*), an *EAC* has use in *QC* if $\dim \mathcal{C} > 1$.
The Hamiltonian H_{SE} has the form

$$H_{SE} = H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + H_I \;, \tag{3}$$

where H_S (H_E) is an hermitian operator on $\mathcal{H}_S$ (respectively, $\mathcal{H}_E$) and H_I, hermitian, acts, for an arbitrary state, in a non trivial way on both factors of the tensor product space $\mathcal{H}_{SE}$. The following lemma states a sufficient and necessary condition for *EAC*'s:
LEMMA 3.1. *A linear subspace $\mathcal{C} \subset \mathcal{H}_S$ is an EAC iff*

i) $H_S \in \mathcal{A}(\mathcal{C})$,
ii) $H_I|_{\mathcal{C}} = \mathbb{I}_{\mathcal{C}} \otimes E(\mathcal{C})$, $(E(\mathcal{C}) = E^\dagger(\mathcal{C}) \in End(\mathcal{H}_E))$.
Proof
We first show that i) and ii) are sufficient conditions. Let $\{|\phi_k\rangle\}$ be an orthonormal set of eigenvectors of the hermitian operator $\tilde{H}_E \doteq H_E + E(\mathcal{C})$, and $\{\tilde{\epsilon}_k\}$ the corresponding set of eigenvalues. Any $\rho_E \in \mathcal{S}(\mathcal{H}_E)$ can written in the form $\rho_E = \sum_{k,h} R_{kh} |\phi_k\rangle\langle\phi_h|$, where R is a hermitian non-negative matrix of rank d_E and trace one with complex matrix elements R_{kh}. For $\rho \in \mathcal{S}(\mathcal{C})$,

$$\begin{aligned} \mathcal{E}_t^{\rho_E}(\rho) &= \mathrm{tr}^E \left(U_{SE}(t)\, \rho \otimes \rho_E\, U_{SE}^\dagger(t) \right) \\ &= \sum_{kh} R_{kh}\, \mathrm{tr}^E \left(e^{-i\,t\,H_S}\, \rho\, e^{i\,t\,H_S} \otimes e^{-i\,t\,(\tilde{\epsilon}_k - \tilde{\epsilon}_h)} |\phi_k\rangle\langle\phi_h| \right) \\ &= e^{-i\,t\,H_S}\, \rho\, e^{i\,t\,H_S} \sum_k R_{kk} = e^{-i\,t\,H_S}\, \rho\, e^{i\,t\,H_S} \;, \end{aligned} \tag{4}$$

in that $\mathrm{tr}^E (|\phi_k\rangle\langle\phi_h|) = \delta_{hk}$, and $\sum_k R_{kk} = 1$.
Suppose now that $\mathcal{C}$ is an EAC. Expanding the identity at point i) of Definition 3.1 up to the first order in t one finds $[\tilde{H}(\rho_E), \rho] = 0$, $\forall \rho \in \mathcal{S}(\mathcal{C})$, where $\tilde{H}(\rho_E) \doteq \mathrm{tr}^E (\rho_E\, H'_{SE})$, and $H'_{SE} \doteq H_{SE} - H_S \otimes \mathbb{I}_E$. From the (manifest) commutativity of $\tilde{H}_{SE}$ with all the states of $\mathcal{C}$ ensues that $\tilde{H}_{SE}|_{\mathcal{C}} = \lambda(\rho_E)\, \mathbb{I}_{\mathcal{C}}$. Moreover, since this property holds for all $\rho_E \in \mathcal{S}(\mathcal{H}_E)$, one has $\langle\phi_i|\, H'_{SE}\, |\phi_i\rangle = \lambda_i\ \mathbb{I}_{\mathcal{C}}$, $\forall |\phi_i\rangle \in \mathcal{H}_E$. It follows from this latter relation that $\langle\phi_i|\, H'_{SE}\, |\phi_{i'}\rangle = \lambda_{ii'}\, \mathbb{I}_{\mathcal{C}}$. Therefore the spectral resolution of H'_{SE} finally reads

$$\begin{aligned} H'_{SE} &= \sum_{jj',ii'} |\psi_j\rangle \otimes |\phi_i\rangle\langle\psi_j|\langle\phi_i|\, H'_{SE}\, |\phi_{i'}\rangle|\psi_{j'}\rangle\langle\psi_{j'}| \otimes \langle\phi_{i'}| \\ &= \sum_j |\psi_j\rangle\langle\psi_j| \otimes \sum_{ii'} \lambda_{ii'} |\phi_i\rangle\langle\phi_{i'}| = \mathbb{I}_{\mathcal{C}} \otimes E \;, \end{aligned} \tag{5}$$

for some $E \in \mathrm{End}(\mathcal{H}_E)$. Here $\{|\psi_j\rangle\}_{j=1}^{d_{\mathcal{C}}}$ $(\{|\phi_i\rangle\}_{i=1}^{d_E})$ is a orthonormal basis of $\mathcal{C}$ (respectively, $\mathcal{H}_E$). The *r.h.s.* of eq. (5) shows that $H_{SE} - H_S \otimes \mathbb{I}_E$, restricted to $\mathcal{C}$ acts trivially on the system Hilbert space, as was to be proven. □
Remark 1. Suppose a unitary $U \in \mathrm{End}(\mathcal{H}_S)$ exists such that $\mathrm{Ad}\, U(H_{SE}) \doteq U\, H_{SE}\, U^\dagger$ satisfies the hypothesis of Lemma 3.1 with respect to subspace $\mathcal{C}$. Then $U^\dagger\, \mathcal{C}$ is an EAC.
The physical meaning of Lemma 3.1 is quite transparent: the states over $\mathcal{C}$ do not suffer any decoherence in that they are all affected by the environment in the same way.
The general form of the interaction Hamiltonian H_I is

$$H_I = \sum_{\lambda \in \Lambda} S_\lambda \otimes E_\lambda \;, \tag{6}$$

where $X_\lambda \in \mathrm{End}(\mathcal{H}_X), X = S, E$, and Λ is a suitable (finite) index set. The operators $\{S_\lambda\}$ will be referred to as *error generators*. Lemma 3.1 basically asserts that $\mathcal{C}$ is an EAC iff $H_S \in \mathcal{A}(\mathcal{C})$ and the S_λ's belong to the subalgebra $\mathcal{A}_1(\mathcal{C}) \subset \mathcal{A}(\mathcal{C})$ of operators with restriction to $\mathcal{C}$ proportional to the identity. Notice that $\mathcal{A}_1(\mathcal{C})$ contains the ideal $\mathcal{A}_0(\mathcal{C})$ of those operators in $\mathcal{A}_1(\mathcal{C})$ which annihilate $\mathcal{C}$. If the error generators belong to $\mathcal{A}_0(\mathcal{C})$ the dynamics on $\mathcal{C} \otimes \mathcal{H}_E$ coincides with that generated by the free Hamiltonian.

The simplest case in which Lemma 3.1 provides an EAC is described in the following

THEOREM 3.1. *Let $\{S_\lambda\}_{\lambda\in\Lambda}$ and H_S form a commutative family of hermitian operators. If $\mathcal{C}$ is a maximal common eigenspace of the S_λ's, then $\mathcal{C}$ is an EAC.*

Proof

Let σ_λ, $(\lambda \in \Lambda)$ be the set of S_λ-eigenvalues, then one has $H_I|_\mathcal{C} = \sum_{\lambda\in\Lambda} \sigma_\lambda \mathbb{I}_\mathcal{C} \otimes E_\lambda = \mathbb{I}_\mathcal{C} \otimes \sum_{\lambda\in\Lambda} \sigma_\lambda E_\lambda \doteq \mathbb{I}_\mathcal{C} \otimes E(\mathcal{C})$. Since $\mathcal{C}$ is maximal and H_S commutes with the S_λ's, then $H_S \in \mathcal{A}(\mathcal{C})$, and the thesis follows from Lemma 3.1. $\Box$

Let now $\mathcal{G}$ be a group, Φ a unitary representation of $\mathcal{G}$ on $\mathcal{H}_S$. $\mathcal{H}_S$, considered as a $\mathcal{G}$-module, has the decomposition, in terms of irreducible $\mathcal{G}$-submodules

$$\mathcal{H}_S = \bigoplus_{j\in\mathcal{J}} n_j \,\mathcal{H}_j \ , \tag{7}$$

where $\mathcal{J}$ is a label set for the $\mathcal{G}$-irreps, $\{\mathcal{H}_j\}_{j\in\mathcal{J}}$ is the set of irreducible submodules of $\mathcal{G}$, and the integers $\{n_j\}_{j\in\mathcal{J}}$ are the corresponding multiplicities. Suppose $\exists j_0 \in \mathcal{J}$ such that $n_{j_0}(\Phi) = 1$, and let $\mathcal{C}$ be the corresponding submodule; then

THEOREM 3.2. *If the $\{S_\lambda\}$'s in equation (6) are $\mathrm{Ad}\,\Phi(\mathcal{G})$-invariant and $H_S \in \mathcal{A}(\mathcal{C})$, then $\mathcal{C}$ is an EAC.*

Proof

Since the S_λ's transform according to the identity representation of $\mathcal{G}$, they can couple only submodules corresponding to equivalent representations. Therefore it follows from $n_{j_0}(\Phi) = 1$ that $S_\lambda \in \mathcal{A}(\mathcal{C})\,(\lambda \in \Lambda)$. Hence the S_λ's commute with all operators of the $\mathcal{G}$-irrep labelled by j_0, and one obtains – from Schur's lemma – that $S_\lambda|_\mathcal{C} \sim \mathbb{I}_\mathcal{C}\,(\lambda \in \Lambda)$. The thesis follows from Lemma 3.1. $\Box$

Let us suppose now that the error generators belong to some representation $\phi\colon \mathcal{A}_S \to \mathrm{gl}(\mathcal{H}_S)$ of a Lie algebra $\mathcal{A}_S$ (*dynamical algebra*). ϕ turns $\mathcal{H}_S$ into an $\mathcal{A}_S$-module that has a decomposition analogous to equation (7) ($\mathcal{J}$ being now a label set for the $\mathcal{A}_S$-irreps).

THEOREM 3.3. *Let $\mathcal{C}$ be the direct sum over a maximal set of equivalent one-dimensional $\mathcal{A}_S$-submodules. Suppose $\mathcal{C} \neq \{0\}$ and $H_S \in \mathcal{A}(\mathcal{C})$; then $\mathcal{C}$ is an EAC.*

Proof

Since $\mathcal{C}$ is spanned by $\mathcal{A}_S$-singlets and $\{S_\lambda\} \subset \phi(\mathcal{A}_S)$ one has $S_\lambda\,|\psi\rangle = \sigma_\lambda\,|\psi\rangle$,

where the σ_λ are c-numbers. Therefore the assumption of Lemma 3.1 holds with $E(\mathcal{C}) = \sum_{\lambda \in \Lambda} \sigma_\lambda \, E_\lambda$. $\square$

Remark 1. When $\mathcal{A}_S$ is semisimple, then the σ_λ's are necessarily zero, and all the one-dimensional irreps are equivalent.

Remark 2. When $\mathcal{A}_S$ is abelian all the irreps are one-dimensional. The subspaces corresponding to the direct sum over a maximal set of equivalent irreps are *weight* spaces.

Remark 3. Theorem 3.3 still holds if the error generators belong to $\phi(\mathcal{U}(\mathcal{A}_S))$, where $\mathcal{U}(\mathcal{A}_S)$ denotes the *universal enveloping algebra* of $\mathcal{A}_S$.

The Lie-algebra representation ϕ is *compatible* (*i.e.* $\mathrm{Ad}\,\Phi(\mathcal{G})$-invariant) with the action of the group $\mathcal{G}$ iff $\Phi(g)\, X\, \Phi^\dagger(g) = X, \, \forall g \in \mathcal{G}, X \in \phi(\mathcal{A}_S)$. In this case, when $\mathcal{A}_S$ is semisimple, the multiplicities of the $\mathcal{A}_S$-irreps ($\mathcal{G}$-irreps) appearing in the decomposition of ϕ (Φ) are but the dimensions of the $\mathcal{G}$-irreps ($\mathcal{A}_S$-irreps) entering the decomposition of Φ (ϕ). In particular this means that the subspace $\mathcal{C}$ obtained as direct sum over the one-dimensional $\mathcal{A}_S$-submodules of ϕ (*singlet sector*) appearing in the decompostion of ϕ is a $\mathcal{G}$-module which enters with multiplicity one in the decomposition of Φ.

THEOREM 3.4. *Let $\mathcal{C}$ be the singlet sector of the $\mathcal{G}$-compatible Lie-algebra representation ϕ of $\mathcal{A}_S$. If*
i) the error generators are $\mathrm{Ad}\,\Phi(\mathcal{G})$*-invariant,*
ii) $H_S \in \mathcal{A}(\mathcal{C})$,
then $\mathcal{C}$ is an EAC.

Proof

The singlet sector corresponds to a $\mathcal{G}$-irrep appearing in the Φ decomposition with multiplicity one. The thesis follows from Theorem 3.2. $\square$

Remark 1. $\phi(\mathcal{U}(\mathcal{A}_S))$ is $\mathrm{Ad}\,\Phi(\mathcal{G})$-invariant in that it is generated by $\mathbb{I}$ and $\phi(\mathcal{A}_S)$.

Remark 2. If $[H_S,\, \phi(\mathcal{A}_S)] = 0$ or $H_S \in \phi(\mathcal{U}(\mathcal{A}_S))$, the condition $H_S \in \mathcal{A}(\mathcal{C})$ is fulfilled.

4. Quantum Registers

In this section the physically relevant notion of *register* is introduced, in analogy with the case of classical computation.

DEFINITION 4.1. *A d-dimensional (quantum) cell C is a quantum system associated to a Hilbert space $\mathcal{H}_C \cong \mathbb{C}^d$. A (quantum) register with N cells is a quantum system given by N replicas of C. R is associated with $\mathcal{H}_R = {\mathcal{H}_C}^{\otimes N}$.*

The register self-hamiltonian will be denoted as H_R. The register Hilbert space $\mathcal{H}_R$ is a natural $\mathcal{S}_N$-module. Let $\{|\psi_j\rangle\}_{j=1}^d$ be a basis of $\mathcal{H}_C$; one can define $\sigma \cdot \otimes_{k=1}^N |\psi_{j_k}\rangle = \otimes_{k=1}^N |\psi_{j_{\sigma(k)}}\rangle$, $(\forall \sigma \in \mathcal{S}_N)$. The latter formula defines, by linear

extension, a representation Φ_N of $\mathcal{S}_N$ on $\mathcal{H}_R$. The operators compatible with this $\mathcal{S}_N$-action lie in the symmetric subspace of $\mathrm{End}\,(\mathcal{H}_R) \cong \mathrm{End}^{\otimes N}\,\mathcal{H}_C$. If each cell of R is coupled with the (common) environment E by a $\mathcal{S}_N$-invariant interaction, one has

$$H_I = \sum_{i=1}^{N} \sum_{\lambda\in\Lambda} S_\lambda^{(i)} \otimes E_\lambda \in \mathrm{End}\,(\mathcal{H}_R \otimes \mathcal{H}_E)\ , \tag{8}$$

where $S_\lambda^{(i)} \in \mathrm{End}\,(\mathcal{H}_R)$, $(i = 1, \ldots, N,\ \lambda \in \Lambda)$ acts as S_λ in the i-th factor of the tensor product $\mathcal{H}_R$, and as the identity in the other factors.
Notice that the register-environment interaction (8) involves only the *coproduct* operators $\Delta^{(N)}(S_\lambda) \doteq \sum_{i=1}^N S_\lambda^{(i)}$. If $\phi\colon \mathcal{A}_S \to \mathrm{gl}(\mathcal{H}_C)$ is a representation of the Lie algebra $\mathcal{A}_S$ in $\mathcal{H}_C$, then $\Delta^{(N)} \circ \phi\colon \mathcal{A}_S \to \mathrm{gl}(\mathcal{H}_R)$ is the N-fold tensor product of ϕ and will be denoted as ϕ_N. An important role in physical applications is played by the case in which $\mathcal{A}_S \cong \mathrm{sl}(d, \mathbb{C})$ and ϕ is the defining representation.
THEOREM 4.1 *Let the quantum register R be coupled with the environment E by the Hamiltonian given by equation (8), where the interaction operators S_λ belong to the defining representation $\tilde{\phi}$ of $\mathrm{sl}(d, \mathbb{C})$ in $\mathcal{H}_C$. Let $\mathcal{C}_N$ be the singlet sector of $\tilde{\phi}_N$. If $H_R \in \mathcal{A}(\mathcal{C}_N)$ then $\mathcal{C}_N$ is an EAC.*
Proof
¿From Theorem 3.4, letting $\mathcal{A}_S = \mathrm{sl}(d,\, \mathbb{C})$, $\mathcal{G} = \mathcal{S}_N$, $\phi = \tilde{\phi}_N$, and $\Phi = \Phi_N$. $\square$
Remark 1. Remarks 1. and 2. of Theorem 3.4 imply immediately that in the latter proposition the error generators are allowed to belong to $\tilde{\phi}_N(\mathcal{U}(\mathrm{sl}(d,\, \mathbb{C})))$ as well. In this case the latter subspace coincides with the whole space of $\mathcal{S}_N$-invariant operators.

5. Conclusions

In this paper we introduced the notion of Error Avoiding Quantum Code as the subspace $\mathcal{C}$ of the Hilbert space of an open quantum system S enbedded in an environment E, in which quantum coherence is preserved. Formally this means that the dynamical (one-parameter) semigroup of S restricted to initial data in $\mathcal{S}(\mathcal{C})$ is given by a (one-parameter) group of unitary transformations. We proved a number of theorems which relate the existence of an EAC to the (dynamical) algebraic structure of the interaction Hamiltonian coupling S and E. In particular we discussed the case of a quantum register symmetrically coupled with the environment. ¿From the broader point of view of the theory of open quantum systems, our results provide a

systematic way of building non-trivial models in which, under quite generic assumptions, the unitary evolution of a subspace is allowed, even while the remaining part of the Hilbert space gets strongly entangled with the environment.

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References

[1] For reviews, see D.P. DiVincenzo, *Science* **270**, 255 (1995);
A. Ekert, and R. Josza, *Revs. Mod. Phys.* **68**, 733, (1996)

[2] H. Bernstein, and U. Vazirani, *Proc. 25-th ACM Symposium on the Theory of Computation*, 1993; p. 11

[3] D. Deutsch, *Proc. Royal Soc. London* **A 425**, 73 (1989)

[4] P.W. Shor, *Phys. Rev. A* **52**, 2493 (1995);
A. Ekert, and C. Macchiavello, *Phys. Rev. Lett.* **77**, 2585 (1996);
D. Gottesman, *Phys. Rev. A* **52**, 1862 (1996);
A.R. Calderbank, E. M. Rains, P.M. Shor, and N.J. Sloane, Phys. Rev. Lett. **78**, 405 (1997)

[5] K. Kraus, "States, Effects, and Operations: Fundamental Notions of Quantum Theory", *Lecture Notes in Physics*, **190**, Springer, Berlin (1983)

[6] P. Zanardi, and M. Rasetti, *Phys. Rev. Lett.* **79**, 3306 (1997)

[7] G. Lindblad, *Commun. Math. Phys.* **48**, 119 (1976)

[8] A. Peres, "Quantum Theory: Concepts and Methods", Kluwer, Dordrecht (1993).